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Dynamical-systems analyses of low-dimensional cloud models with emphasis on cirrus and contrails

Executive summary

The strongest **cirrus-specific** examples I found are the reduced subvisible-cirrus model of **Spreitzer, Marschalik, and Spichtinger** and the newer pure-ice-cloud oscillator model of **Bergner and Spichtinger**. Both collapse cloud microphysics into **three autonomous ODEs** and then use **phase-space analysis, Jacobian eigenvalues, Hopf-bifurcation diagnosis, and numerical integration** to expose a clean physical picture: cloud evolution can switch from a **stable focus** with damped oscillations to a **self-sustained limit cycle**, driven by the competition between supersaturation buildup, ice nucleation/growth, and sedimentation. In the newer paper, the oscillatory regime is even bounded by **two supercritical Hopf bifurcations**, which yields a particularly clear "oscillation window" in parameter space. [1]

The best-developed **cloud-macrophysical** dynamical-systems literature is not about cirrus or contrails, but about **marine stratocumulus cloud–rain interaction**. The canonical line runs from **Koren & Feingold (2011)** to **Koren, Tziperman & Feingold (2017)** and then to mathematically deeper follow-ups such as **Chekroun, Koren & Liu (2020)** and **Liu et al. (2022)**. These papers use a **delay differential equation** or related low-order systems to show how cloud depth, droplet concentration, rain delay, and recovery time generate **steady states, limit cycles, hysteresis-consistent behavior, period-doubling, double-Hopf structure, and in some regimes chaos**. Physically, the first Hopf point marks the onset of recurrent cloud deepening and rain depletion cycles; aerosol loading generally acts as a stabilizer by delaying or weakening rain formation. [2]

A key practical conclusion is that **formal continuation software is uncommon in the papers most relevant to cirrus**. The cirrus papers I found mainly identify bifurcations by **parameter scanning, equilibrium finding, Jacobian eigenvalue tracking, and Poincaré-section computations**, rather than by explicit AUTO or MATCONT workflows. In the warm-cloud delay literature, analytical reductions and bifurcation diagnosis are more prominent than software-heavy continuation. That means the field already contains useful dynamical insight, but there is still room for a more systematic continuation-based program, especially for cirrus and even more so for contrails. [3]

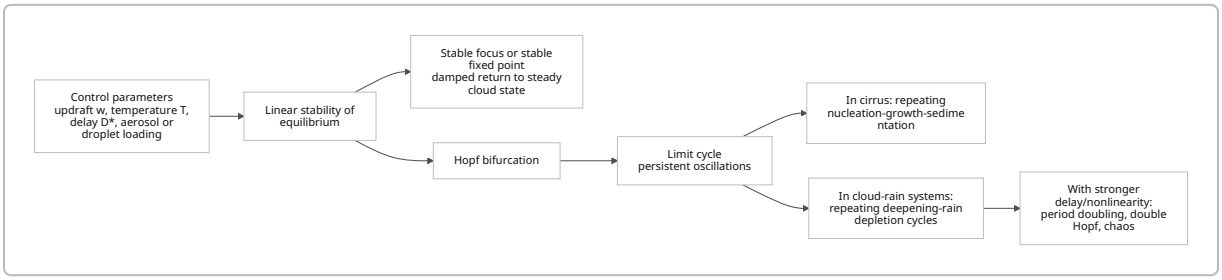
For **aircraft contrails**, I did **not** find a primary literature tradition that applies continuation, center-manifold reduction, normal forms, or phase-space bifurcation analysis to low-dimensional contrail models. The reduced-order contrail literature is instead dominated by **0D box models, multi-trajectory/offline microphysics, Gaussian-plume life-cycle models, and newer intermediate-complexity PDE/jet frameworks**. Those models are valuable and physically reduced, but they are generally not analyzed as nonlinear dynamical systems in the sense you asked for. [4]

What the literature actually contains

Two distinct clusters dominate this niche. The first is a small but important **ice-cloud/cirrus** cluster built around simplified parcel or box models with variables such as **ice crystal number concentration, ice mass concentration, and supersaturation or relative humidity**. The second is a richer **warm-cloud/aerosol–cloud–rain** cluster where cloud thickness, droplet concentration, and delayed rain production are reduced to a few state variables or a scalar DDE. The second cluster supplies much of the mathematical machinery; the first cluster gives the cirrus-specific physical interpretation you asked for. [5]

Within the cirrus literature, the strongest papers show that low-dimensional models can reveal **regime boundaries** that are hard to see in full microphysical simulations: for example, when a cloud asymptotically settles into a thin persistent state versus when it enters a persistent nucleation–growth–sedimentation cycle. Within the warm-cloud literature, the same toolkit clarifies when cloud systems behave like **damped oscillators**, **self-sustained oscillators**, or even systems undergoing more exotic delay-driven bifurcations. [6]

Contrails are the conspicuous gap. Reduced models absolutely exist, and some are quite elegant, but the research objective has mostly been **prediction of formation, microphysics, or life-cycle metrics**, not nonlinear phase-space structure. The contrast is sharp: warm-cloud and cirrus studies explicitly discuss equilibria, eigenvalues, Hopf points, limit cycles, and phase portraits, whereas contrail papers usually discuss **plume dilution, wake phases, Gaussian spreading, or trajectory ensembles**. [7]



This schematic is a synthesis of the cirrus and cloud–rain papers discussed below; the specific branches and parameter names differ by model, but the common organizing principle is the same. [6]

Selected papers compared

Paper	Model and dimensionality	Governing variables and key parameters	Dynamical methods	Main regimes and interpretation	Reproducibility and source
Spreitzer, Marschalik & Spichtinger, 2017	**3D ODE** reduced model for **subvisible cirrus**	Variables: ice crystal number concentration N_c, ice mass concentration q_c, relative humidity over ice RHi; key controls: **temperature** and **vertical updraft** w	Numerical integration, equilibrium finding, Jacobian eigenvalues, phase-space trajectories, Poincaré map, Hopf diagnosis	**Stable focus** versus **limit cycle**. Hopf line in (w, T) separates damped approach to persistent thin cloud from undamped oscillatory cloud evolution. Physical reading: interplay of homogeneous nucleation, diffusional growth, and sedimentation can sustain oscillations, so homogeneous freezing remains a viable pathway for subvisible cirrus.	Open-access article/PDF. I did not see AUTO or MATCONT mentioned in the text. Data are described in the paper, but no public code repository is obvious from the article. [8]

Paper	Model and dimensionality	Governing variables and key parameters	Dynamical methods	Main regimes and interpretation	Reproducibility and source
Bergner & Spichtinger, 2026	**3D ODE** model for **pure ice clouds**	Variables: ice number n, ice mass q, saturation ratio over ice s; key controls: **pressure** p, **temperature** T, **vertical velocity** w, **sedimentation parameter** F	Linearization, numerical eigenvalue scans, semi-analytical bifurcation fits, limit-cycle period estimation, scaling analysis	Two **supercritical Hopf bifurcations** bound an unstable window where a limit cycle exists; outside that window the unique equilibrium is stable. Physical reading: there is an intermediate updraft regime where ice clouds behave as nonlinear oscillators, while too-weak or too-strong forcing restores steady behavior; bifurcation curves scale approximately like $\sqrt{F}$.	Peer-reviewed Chaos article with open arXiv preprint. I did not find a separate code/data repository in the preprint text. [9]

Paper	Model and dimensionality	Governing variables and key parameters	Dynamical methods	Main regimes and interpretation	Reproducibility and source
Koren & Feingold, 2011	Low-order **delay-coupled predator–prey-like** cloud model; effectively **3 variables** (H, R, N_d) with delay	Cloud depth H, rain rate R, droplet concentration N_d; key controls: carrying capacities H_0, N_0, recovery times τ_1, τ_2, delay T_0	Analytical steady states, time-dependent simulations, phase diagrams	Weak rain gives **damped oscillations to steady state**; stronger drizzle and larger delay can produce **steady oscillations / limit-cycle behavior**; very strong precipitation yields an unstable/ collapse regime. Physical reading: rain is the predator, cloud the prey, aerosol the modulator; aerosol loading acts as a damping/ stabilizing influence by suppressing rain formation.	Accessible PDF mirror and abstract, but no public code/data package is evident from accessible text. [10]

Paper	Model and dimensionality	Governing variables and key parameters	Dynamical methods	Main regimes and interpretation	Reproducibility and source
Feingold & Koren, 2013	Network of **coupled low-order oscillators** for aerosol–cloud–precipitation elements	Individual cloud elements inherit predator–prey-like dynamics; controls include **coupling strength** and **communication delay**	Coupled-oscillator analysis with numerical experiments	Weak coupling gives quasi-independent oscillators; stronger coupling can produce **synchronization** and some stabilization; still stronger coupling can destabilize growth. Physical reading: the model mimics organization and synchronization of **open and closed cells** seen in satellite imagery and LES.	Open-access article/PDF; no explicit continuation package reported on article page. [11]

Paper	Model and dimensionality	Governing variables and key parameters	Dynamical methods	Main regimes and interpretation	Reproducibility and source
Koren, Tziperman & Feingold, 2017	Scalar **DDE** for cloud depth h after nondimensionalization	$dh/dt^* = 1 - h - \mu^{-1}h^2(t^* - D^*)$; key nondimensional controls: μ and **delay** D^*, with $\mu = \sqrt{N}/(\alpha s H_0)$	Linear stability analysis around fixed point, Lambert W solution for characteristic exponent, analytical Hopf condition, numerical simulations	Overdamped regime, damped oscillations, **supercritical Hopf** to a limit cycle, and **period-doubling route to chaos** in a nonphysical regime. Physical reading: increased delay destabilizes; larger droplet concentration N or stronger recovery stabilizes. The paper also makes explicit why one low-dimensional parameter μ combines aerosol and environmental controls.	Accessible PDF; no code/ data repository evident in the text. [12]

Paper	Model and dimensionality	Governing variables and key parameters	Dynamical methods	Main regimes and interpretation	Reproducibility and source
Chekroun, Koren & Liu, 2020	Reduced **DDE cloud–rain** models	Koren–Feingold-type delay models	**Galerkin–Koornwinder reduction**, **center-manifold/ Stuart–Landau normal-form** diagnosis of Hopf, bifurcation classification	Shows both **supercritical and subcritical Hopf** regimes in cloud–rain models; identifies **double-Hopf bifurcations** and maps where different local behaviors occur. Physical reading: seemingly minor changes in nonlinear delay structure can flip the cloud–rain cycle from soft onset to hard onset with finite-amplitude oscillations.	Primary article appears paywalled; abstract is informative, but I did not find a public code archive from the accessible sources. [13]

Paper	Model and dimensionality	Governing variables and key parameters	Dynamical methods	Main regimes and interpretation	Reproducibility and source
Liu, Pei, Wang & Qi, 2022	Generalized delayed **Koren–Feingold cloud–rain** system	Time delay in rain formation is the distinguished bifurcation parameter	Stability analysis, **double-Hopf bifurcation analysis**, chaos diagnostics	Time delay controls transitions from stable equilibrium to oscillatory and more complex dynamics; the paper explicitly proves **double-Hopf** occurrence. Physical reading: rain-conversion delay can create competing oscillatory modes and greater tipping sensitivity in cloud–rain dynamics.	Abstract and publisher PDF landing page accessible; no public code/ data repository was evident from those sources. [14]

Paper	Model and dimensionality	Governing variables and key parameters	Dynamical methods	Main regimes and interpretation	Reproducibility and source
Wacker, 2006	**4 ODE** two-layer conceptual cloud model	Variables C_1, P_1, C_2, P_2 for cloud-water and precipitation in two layers; key controls: **upwind velocity** w and external source rates Φ	Structural analysis, bifurcation diagrams, linear stability, phase-space trajectories	For weak upwind, the unique steady state is **unstable** and the attractor is a **limit cycle**; for intermediate w, **multiple steady states** exist; stronger forcing can eliminate cusps and leave a unique attractor. Physical reading: advection can suppress precipitation aloft, and the authors explicitly compare one branch to **cirrus uncinus** cloud heads and fallstreaks.	Open-access PDF. The paper notes that empirical data for systematic verification of its bifurcation diagrams were not available. [15]

Cirrus and ice-cloud case studies

The **Spreitzer et al.** model remains the clearest cirrus-specific demonstration of how a reduced cloud-microphysics model gains explanatory power from dynamical-systems analysis. After simplifying the governing processes to autonomous ODEs for N_c, q_c, and RHi, the authors scan 190–$230\ \mathrm{K}$ and $0 < w \leq 0.05\ \mathrm{m\ s^{-1}}$, compute the equilibrium, linearize around it, and classify the eigenstructure. They find a unique equilibrium whose Jacobian has one always-negative real eigenvalue and a complex-conjugate pair whose real part changes sign with w and T. That gives a physically crisp story: the same microphysics can either produce a **persistent subvisible cloud** reached through damped oscillations or a **limit-cycle cloud** with continuing oscillations in humidity and ice content. [16]

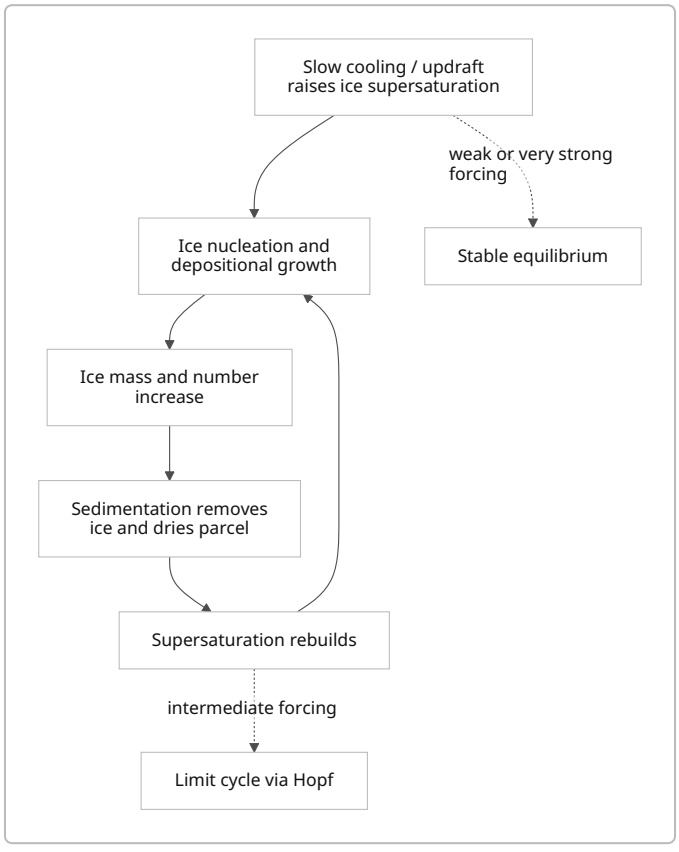
The governing reduced system is especially interpretable because each equation can be read as a bookkeeping balance between a small number of physical tendencies: nucleation versus sedimentation for number concentration, nucleation plus growth minus sedimentation for mass concentration, and cooling-driven supersaturation versus depositional drawdown for humidity. That compactness is exactly why the Hopf point has physical meaning rather than being a purely mathematical ornament. The oscillatory mode

amounts to a repeating cycle of **supersaturation buildup → nucleation/growth burst → sedimentation/growth sink → renewed buildup**. [17]

The newer **Bergner & Spichtinger** paper is, in my view, the most analytically satisfying example in the ice-cloud literature. It keeps the state space at **three dimensions** but pushes the math further: equilibrium existence/uniqueness, numerical eigenvalue scans, two **supercritical Hopf bifurcations**, semi-analytical fits to bifurcation curves, and scaling of those curves with the sedimentation efficiency parameter F. One striking result is that the oscillatory regime is not just "beyond one threshold"; it sits **between two thresholds** in T–w space, so both too-weak and too-strong forcing can return the system to a stable equilibrium. [18]

Just as important, they interpret one eigenvalue as a **strongly contracting direction**, which implies quasi-two-dimensional long-time dynamics, and they note that a **Stuart–Landau normal form** might be a natural next reduction. That makes this paper unusually valuable for future continuation or center-manifold work: it already hints at the reduced manifold geometry rather than only cataloging the bifurcation set. [19]

A conceptual summary of the cirrus/ice-cloud picture is:



That cycle is not just an analogy; it is the mechanism the reduced models encode mathematically. In **Spreitzer et al.**, it explains the transition from stable focus to limit cycle for subvisible cirrus. In **Bergner & Spichtinger**, it becomes a broader oscillator framework for pure ice clouds with explicit scaling laws. [20]

Cloud-rain and multilayer cloud exemplars

The **Koren–Feingold** line of work is the clearest non-cirrus demonstration that low-order cloud models can produce physically legible bifurcation structure. In the 2011 predator–prey formulation, the three evolving quantities are **cloud depth** H, **rain rate** R, and **droplet concentration** N_d, regulated by carrying capacities, recovery times, and a rain-production delay. The paper shows three regimes: **weakly precipitating damped return to steady state**, **stable oscillations/limit cycles**, and a strong-precipitation regime associated with boundary-layer collapse. The key physical takeaway is that **aerosol loading stabilizes the system** by weakening rain production, while increased delay favors oscillations by allowing cloud depth to overshoot before rain responds. [21]

The 2017 **Exploring the nonlinear cloud and rain equation** paper makes the bifurcation structure much sharper. By nondimensionalizing the cloud-depth DDE to a scalar equation with parameters μ and D^*, the authors reduce three environmental controls and droplet concentration to a single composite parameter μ. Linearization about the fixed point, together with the **Lambert W** representation of the characteristic exponent, yields an explicit analytical Hopf condition. The result is a particularly transparent phase diagram: overdamped decay, damped oscillations, a **supercritical Hopf transition**, and, farther into a nonphysical regime, **period doubling to chaos**. [22]

What makes this model physically insightful is not just that it oscillates, but **why**. Delay is a stand-in for the finite time needed to convert cloud water into rain, while μ packages the balance between aerosol-induced droplet loading and environmental cloud recovery. When delay increases or effective rain production becomes too strong, the cloud deepens “too far” before the sink turns on, and the system loses steady balance. That is an atmospheric interpretation of a textbook Hopf, and it remains one of the most teachable examples in the literature. [23]

The mathematically deeper follow-up by **Chekroun et al. (2020)** shows that the cloud–rain DDE is richer than the first Hopf diagram suggests. Using a **Galerkin–Koornwinder reduction** to the **Stuart–Landau normal form** and center-manifold setting, the paper diagnoses where the Hopf bifurcation is **supercritical** versus **subcritical**, and reports **double-Hopf bifurcations** that bound “islands” of one type inside regions of another. This matters physically because subcritical onset implies finite-amplitude oscillations and stronger tipping behavior than the soft onset associated with a supercritical Hopf. [24]

The 2022 **Liu et al.** study pushes that delay-centric interpretation further by proving **double-Hopf** structure in a generalized Koren–Feingold system and tying the dynamical transitions directly to the rain-production delay. In practical cloud terms, that means the timescale of warm-rain conversion is not just a nuisance parameter; it can qualitatively reorganize the solution space. [14]

A different but still useful classic is **Wacker (2006)**. This is a **four-dimensional** two-layer cloud model with cloud water and precipitation in each layer. Its bifurcation diagrams show that varying **upwind velocity** w can generate **an unstable single steady state with a limit-cycle attractor**, then a band of **multiple steady states**, and later a unique steady state again. The paper also finds **cusps** in the steady-branch structure as external source parameters vary. Although it is not a cirrus microphysics paper in the modern sense, it is valuable because it explicitly maps mathematical branches to familiar cloud archetypes, including a **cirrus uncinus-like** state where upward transport suppresses precipitation in the updraft head and favors fallstreak formation downstream. [25]

Separate LES work on closed-to-open-cell transitions also shows that the real cloud system can display **hysteresis** and asymmetry in reversal, which supports the physical plausibility of multiple low-order attractors or history dependence. The 2017 DDE paper explicitly notes this connection, and later reviews summarize the open/closed-cell transition as a bistability problem. [26]

Contrails and what is missing

The contrail literature has many reduced models, but they are usually reduced in the sense of **computational complexity**, not in the sense of **explicit dynamical-systems reduction**. A good snapshot is the new **RadMod** work, which describes current practice as spanning expensive **3D LES/RANS models** and inexpensive **0D box models**. RadMod itself inserts a **2D advection–diffusion** dynamical core between those extremes to model jet evolution, mixing, temperature, and water vapor while remaining cheaper than full CFD. [27]

That paper is also explicit about what earlier "low-dimensional" contrail models do: they often compute microphysics along **prescribed dilution trajectories** or with a **multi-0D** ensemble of box-model runs. Those approaches are excellent for sensitivity studies but are not framed as bifurcation problems, and the article motivates RadMod precisely because offline box approaches miss physically important mixing interactions among nearby trajectories. [27]

Likewise, life-cycle contrail models such as the 2026 **overlapping-contrails** study organize development into **jet, vortex, dissipation, and diffusion** phases and use extensions of a **Gaussian plume** framework with wake-vortex parameterizations to estimate persistence. This is reduced-order and physically interpretable, but again the analysis target is contrail lifetime and geometry, not equilibria, eigenvalues, or phase-space transitions. [28]

Recent comparison work underscores the distinction. A 2025 ACP article shows that **zero-dimensional contrail models** can substantially underpredict integrated extinction relative to a higher-fidelity model and even show opposite humidity sensitivity. That is exactly the kind of discrepancy for which dynamical-systems analysis could, in principle, become valuable: one would like to know whether some of these reduced models hide thresholds, bistability, or stiffness-related transitions. But as of the literature I found, that program has not really been carried out. [29]

So the most honest answer on contrails is: **there are low-dimensional contrail models, but not yet a mature contrail bifurcation literature**. The natural opportunities would be to analyze persistent-versus-evanescent contrails, wake-to-diffusion transition timing, supersaturation-driven extinction thresholds, and humidity–mixing feedbacks using the same toolkit that has already proved useful for cloud–rain systems and simplified cirrus. [30]

Reproducibility and methodological takeaways

Reproducibility is mixed. Several of the most useful papers are **open access** or have an accessible PDF: **Spreitzer et al. 2017**, **Wacker 2006**, **Feingold & Koren 2013**, and the 2017 **cloud-and-rain** paper. **Bergner & Spichtinger** also have an open arXiv preprint alongside the peer-reviewed 2026 Chaos publication. That is very good for inspection and reuse of equations. [31]

What is **not** common is modern research-compendium practice. In the cirrus and cloud–rain papers I inspected, I did not find obvious public **GitHub/Zenodo code repositories** or packaged datasets linked from the article text. Spreitzer et al. include a brief data-availability note saying the data used are described in the paper; the Bergner preprint does not show a separate data/code availability section in the text I searched; and the 2017 cloud–rain paper shows no code or data statement in the PDF text. [32]

Methodologically, the most robust pattern across papers is:

1. **Write the smallest model that preserves the feedback loop of interest**: nucleation–growth–sedimentation for cirrus, or cloud-growth–rain-delay–aerosol replenishment for marine low clouds. [33]
2. **Find equilibria and classify them by eigenvalues**. In this literature, that alone already separates physically meaningful regimes. [34]
3. **Use Hopf bifurcations as the main interpretive event**. In nearly every successful example, Hopf corresponds to the onset or cessation of repeating physical cycles. [35]
4. **Go beyond local stability only when the reduced model warrants it**. The best extensions are Poincaré maps for limit cycles, Lambert W for DDE spectra, and Stuart–Landau/center-manifold reductions when the cloud model truly behaves like a low-order oscillator. [36]

The main limitation, especially for your cirrus/contrail focus, is that the literature is still skewed toward **bifurcation-by-parameter-scan** rather than full-fledged continuation of equilibria and periodic orbits. The immediate research opportunity is therefore clear: take the already persuasive reduced cirrus models and run a modern continuation workflow on them, then do the same for the most promising contrail box/plume models. The underlying physics is simple enough, and the payoff in interpretability could be large. [37]

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